the ray $B(1, 1')$. This ray intersects AC in a point which we call $1''$. Construct the projective scale $0''$, $1''$, $2''$, $3''$, $4''$, $\cdots$ on AC, taking $A = 0''$, $C = \infty''$, and draw the rays $B2''$, $B3''$, $B4''$, $\cdots$. We thus have in our triangle a set of rays from each vertex.[1] Mark in the figure all points in which three rays meet. These points we shall call, for convenience, knot-points. By comparing Fig. 3 with Fig. 1 and Fig. 2, we find:

The number r of knot-points on An, $n = 2, 3, 4, 5, \cdots$, gives the number of distinct divisors of n, counting both 1 and n as divisors. Therefore n is prime, when and only when there are two knot-points on An. When $r = 3$, n is the square of a prime number, and, in a similar way, some other elementary results (for example the determination of common prime factors of given integers) may be read off the figure.—Readers familiar with projective geometry will recognize immediately how Fig. 3 may be derived from Fig. 2.

Other projective geometric constructions for determining the prime-character of a given number are easily devised, but the construction given above has the advantage of involving only straight lines.

When one ignores simple relations like the one treated in this note, between projective geometry and number theory, which in reality are only analogies without deeper import, the problem of finding connections between projective geometry and number theory seems to be, according to an interesting remark by Klein (Elliptische Modulfunktionen, Vol. I, p. 242, footnote), a task of great difficulty. (See also Weber-Wellstein, II, 1907, pp. 211–212, footnote.)

CONCERNING PREFERENTIAL VOTING.

By L. L. DINES, University of Saskatchewan.

In a recent number of the MONTHLY, Professor W. V. Lovitt has given an interesting discussion of a problem which may be stated as follows:

In a certain election, each voter expresses his first, second, and third choice. The three candidates A, B, and C receive respectively A_i, B_i, and C_i votes for ith choice. It is required to determine what weights may be assigned to the votes for first, second, and third choice in order that A may win.

While the purely algebraic discussion given in the present note is less vivid than the geometric discussion of Professor Lovitt, it may be of interest as it leads more directly to the necessary conditions for the existence of a solution, and to explicit ranges of possible values for the weights in terms of the data A_i, B_i, and C_i. Furthermore, the algebraic method can be extended to a treatment of the more general problem in which there are n candidates, and each voter

[1] The $\triangle ABC$ now represents, for purposes of projective measurement, the whole first quadrant in an ordinary system of coördinates. It is interesting to see how, by taking on the sides of our triangle (extending each indefinitely in both directions) the "negative" and the "rational" points into account, the whole ordinary plane is covered by a "projective" plane.

expresses his first, second, $\cdots$, and mth choice. This more general problem is considered briefly in § 2.

§ 1. Let the weights of first, second, and third choice be x_1, x_2, and x_3 respectively, with the condition

$$x_1 > x_2 > x_3 > 0. \tag{1}$$

Then since the number of points received by A is given by $A_1x_1 + A_2x_2 + A_3x_3$, with similar expressions for the number of points received by B and C, it follows that the necessary and sufficient conditions that A shall win are expressed by the inequalities

$$\begin{aligned} &(A_1 - B_1)x_1 + (A_2 - B_2)x_2 + (A_3 - B_3)x_3 > 0, \\ &(A_1 - C_1)x_1 + (A_2 - C_2)x_2 + (A_3 - C_3)x_3 > 0. \end{aligned} \tag{2}$$

These inequalities may be simplified by introducing instead of the weights, the differences between the weights, by the substitutions

$$x_1 - x_2 = \xi_1, \quad x_2 - x_3 = \xi_2, \quad x_3 = \xi_3,$$

or

$$x_1 = \xi_1 + \xi_2 + \xi_3, \quad x_2 = \xi_2 + \xi_3, \quad x_3 = \xi_3,$$

with the condition (1) replaced by the conditions

$$\xi_i > 0 \qquad i = 1, 2, 3. \tag{1'}$$

Expressed in terms of the ξ's, conditions (2) are

$$(A_1 - B_1)(\xi_1 + \xi_2 + \xi_3) + (A_2 - B_2)(\xi_2 + \xi_3) + (A_3 - B_3)\xi_3 > 0,$$

$$(A_1 - C_1)(\xi_1 + \xi_2 + \xi_3) + (A_2 - C_2)(\xi_2 + \xi_3) + (A_3 - C_3)\xi_3 > 0;$$

which upon collecting terms in ξ_1, ξ_2, and ξ_3, and making use of the relations

$$A_1 + A_2 + A_3 = B_1 + B_2 + B_3 = C_1 + C_2 + C_3 = \text{number of voters}, \tag{3}$$

may be written

$$(A_1 - B_1)\xi_1 > (A_3 - B_3)\xi_2 \tag{4}$$

$$(A_1 - C_1)\xi_1 > (A_3 - C_3)\xi_2. \tag{5}$$

Since ξ_3 does not appear in these condition, ξ_3 $(= x_3)$ may have any positive value. Also since (4) and (5) obviously impose a restriction only on the ratio $\xi_1 : \xi_2$, ξ_2 $(= x_2 - x_3)$ may have any positive value. It remains to determine the ranges of values which ξ_1 may have when conditioned by (4) and (5). The nature of these conditions is made more apparent by separating them into cases according to the value of the coefficient of ξ_1, as follows:

(4′) (a) If $A_1 - B_1 > 0,\ \xi_1 > \dfrac{A_3 - B_3}{A_1 - B_1}\xi_2$; (b) If $A_1 - B_1 = 0,\ (A_3 - B_3)\xi_2 < 0$;

(c) If $A_1 - B_1 < 0,\ \xi_1 < \dfrac{A_3 - B_3}{A_1 - B_1}\xi_2$;

(5′) (a) If $A_1 - C_1 > 0,\ \xi_1 > \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$; (b) If $A_1 - C_1 = 0,\ (A_3 - C_3)\xi_2 < 0$;

(c) If $A_1 - C_1 < 0,\ \xi_1 < \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$.

In considering the possible order relations which may exist between the three quantities A_1, B_1, C_1, those which differ from each other only in having B_1 and C_1 interchanged may for our purposes be considered equivalent. With this understanding, the following table which can now be filled out by reference to the conditions (4′), (5′), and (1′), gives the complete solution of our problem for all cases.

Case	Necessary Conditions	Range for ξ_1
$A_1 > B_1 \geqq C_1$	....	$\xi_1 > \left\{ \dfrac{A_3 - B_3}{A_1 - B_1}\xi_2 \text{ and } \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2 \right\}$
$A_1 = B_1 > C_1$	$A_3 < B_3$*	$\xi_1 > \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$
$A_1 = B_1 = C_1$	$A_3 < B_3$*, $A_3 < C_3$*	ξ_1 any positive value
$C_1 > A_1 > B_1$	$\dfrac{A_3 - B_3}{A_1 - B_1} < \dfrac{A_3 - C_3}{A_1 - C_1}$, $A_3 < C_3$	$\dfrac{A_3 - B_3}{A_1 - B_1}\xi_2 < \xi_1 < \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$
$C_1 > A_1 = B_1$	$A_3 < B_3$*, $A_3 < C_3$	$\xi_1 < \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$
$B_1 \geqq C_1 > A_1$	$A_3 < B_3$, $A_3 < C_3$	$\xi_1 < \left\{ \dfrac{A_3 - B_3}{A_1 - B_1}\xi_2 \text{ and } \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2 \right\}$

It may be noted that on account of the relations (3), the conditions $A_3 < B_3$ and $A_3 < C_3$ when marked with a * in the column of necessary conditions, may be replaced respectively by $A_2 > B_2$ and $A_2 > C_2$.

§ 2. Suppose now there are $n + 1$ candidates: A, and $B^{(i)}$ $(i = 1, 2, \cdots, n)$; and the jth choice is to receive the weight x_j $(j = 1, 2, \cdots, m)$, with the condition $x_k > x_{k+1} > 0$. Then the necessary and sufficient conditions that A be elected are

$$(6) \qquad \sum_{j=1}^{m} (A_j - B_j^{(i)})x_j > 0, \qquad i = 1, 2, \cdots, n.$$

If we now introduce as new variables the differences between the weights, by means of the substitutions

$$x_j = \sum_{h=j}^{m} \xi_h, \qquad j = 1, 2, \cdots, m,$$

we obtain in place of (6) the conditions

$$\sum_{j=1}^{m} (A_j - B_j^{(i)}) \sum_{h=j}^{m} \xi_h > 0, \qquad i = 1, 2, \cdots, n,$$

which may be written

$$\sum_{h=1}^{m}\sum_{j=1}^{h}(A_j - B_j^{(i)})\xi_h > 0, \qquad i = 1, 2, \cdots, n. \tag{7}$$

The n conditions of this set may be divided into three sets, according as $A_1 - B_1^{(i)}$, the coefficient of ξ_1, is positive, zero, or negative. Without loss of generality, we may suppose that in the first n_1 inequalities of (7) the leading coefficient is positive, in the next n_2 inequalities it is zero, while in the remaining n_3 $(= n - n_1 - n_2)$ inequalities it is negative. Conditions (7) are then replaced by the following three sets:

Type I, in which $A_1 - B_1^{(i)} > 0$, by

$$\xi_1 > -\sum_{h=2}^{m}\sum_{j=1}^{h}\frac{A_j - B_j^{(i)}}{A_1 - B_1^{(i)}}\xi_h, \qquad i = 1, 2, \cdots, n_1; \tag{I}$$

Type II, in which $A_1 - B_1^{(i)} = 0$, by

$$\sum_{h=2}^{m}\sum_{j=1}^{h}(A_j - B_j^{(i)})\xi_h > 0, \quad i = n_1 + 1, n_1 + 2, \cdots, n_1 + n_2; \tag{II}$$

Type III, in which $A_1 - B_1^{(i)} < 0$, by

$$\xi_1 < -\sum_{h=2}^{m}\sum_{j=1}^{h}\frac{A_j - B_j^{(i)}}{A_1 - B_1^{(i)}}\xi_h, \quad i = n_1 + n_2 + 1, n_1 + n_2 + 2, \cdots, n. \tag{III}$$

Of these three sets it will be noticed that (II) places no restriction on ξ_1, while (I) and (III) state respectively n_1 lower bounds and n_3 upper bounds for ξ_1. Since these upper bounds must all be positive, $\xi_2, \xi_3, \cdots, \xi_m$ must satisfy the n_3 conditions

$$\sum_{h=2}^{m}\sum_{j=1}^{h}(A_j - B_j^{(i)})\xi_h > 0, \quad i = n_1 + n_2 + 1, n_1 + n_2 + 2, \cdots, n. \tag{8}$$

Also in order that there may be an existent range for ξ_1 between the lower bounds of (I) and the upper bounds of (III), the quantities $\xi_2, \xi_3, \cdots, \xi_m$ must also satisfy the $n_1 n_3$ conditions

$$\sum_{h=2}^{m}\sum_{j=1}^{h}\left(\frac{A_j - B_j^{(i)}}{A_1 - B_1^{(i)}} - \frac{A_j - B_j^{(k)}}{A_1 - B_1^{(k)}}\right)\xi_h > 0, \tag{9}$$
$$i = 1, 2, \cdots, n_1,$$
$$k = n_1 + n_2 + 1, n_1 + n_2 + 2, \cdots, n.$$

We now have ξ_1 restricted only by (I) and (III); and the existence of a positive range for ξ_1 satisfying these conditions is assured by (8) and (9). The next step is to consider the conditions imposed upon $\xi_2, \xi_3, \cdots, \xi_m$ by (II), (8) and (9). These form a system of linear, homogeneous inequalities ($n_2 + n_3 + n_1 n_3$ in number), which may now be classified into Types I, II and III, according as the coefficients of ξ_2 are positive, zero, or negative. Those of Types I and III will

determine certain lower and upper bounds for ξ_2, while those of Type II, together with the conditions that the range for ξ_2 shall contain positive values, will constitute a system of linear, homogeneous inequalities in $\xi_3, \xi_4, \cdots, \xi_m$.

The process indicated can now be repeated, the ranges for $\xi_3, \xi_4, \cdots$ being determined successively, each in terms of the ξ's of higher subscript. After the first $(r - 1)$ ξ's have been thus *eliminated*, we are met by a system of conditions of form

$$\sum_{h=r}^{m} a_h^{(l)}\xi_h > 0, \qquad l = 1, 2, \cdots, l_r, \tag{10}$$

where the coefficients $a_h^{(l)}$ are rational functions of the differences $A_j - B_j^{(i)}$ This system (10) may be classified into Types I, II, and III, according as the leading coefficients $a_r^{(l)}$ are positive, zero, or negative.

A necessary and sufficient condition that there exist a system of weighting under which A wins, is that at some stage of the process of successive elimination of the variables ξ_i described above, the inequalities of the system (10) *presenting itself shall be all of Type I.*

To show the necessity of the condition, we suppose that the system of conditions presenting itself at each stage of the elimination contains inequalities of Types II or III. Since the elimination of the ξ of lowest subscript from such a system leads to a system of conditions upon the ξ's of higher subscript, it follows that under our assumption, the successive elimination of the ξ's will lead finally to a system of conditions

$$a_m^{(l)}\xi_m > 0, \quad l = 1, 2, \cdots, l_m,$$

which contains only ξ_m. But according to our assumption not all the coefficients $a_m^{(l)}$ are positive. Hence the system of conditions cannot be satisfied by a positive ξ_m.

If, on the contrary, the inequalities (10) at any stage are all of Type I, they determine only *lower* bounds for ξ_r. The variables $\xi_{r+1}, \xi_{r+1}, \cdots, \xi_m$ can then have any positive values, ξ_r can have any positive values greater than these lower bounds, and positive values can be assigned to $\xi_{r-1}, \xi_{r-2}, \cdots, \xi_1$, in order, satisfying the conditions imposed in the elimination process.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

The Continuum and Other Types of Serial Order. By EDWARD V. HUNTINGTON. Second Edition. Harvard University Press, Cambridge, Mass., 1917.

It is to some extent customary for younger students of our subject to cultivate skill in the use of great mathematical doctrines without perceiving the logical setting, in mathematical and physical science, of the complex and real number systems upon which most of these doctrines depend. /And, indeed, we can all